Dynamic Behaviour in a Bertrand Model With Bounded Rationality

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Abstract
Imperfect competition represents a well-known topic of modern economic analysis. It can be easily noticed in the current economic climate, when manifesting in relation with product price (Bertrand type), quantity (Cournot type) or quality. Whether we are talking about the price or output competition scenario, a common denominator would be the fact that in the first period of the oligopoly theory, the researchers were focused on the static approach of these models. Subsequently, however, an increasing interest in a dynamic approach was noted, with a more detailed analyses being considered to better align with the economic reality. The present paper intents to analyse a Bertrand dynamic duopoly, with players competing in price terms, while making use of a bounded rationality mechanism in order to establish charging levels. Based on a two nonlinear difference equational system, the mathematic modelling of such a game is focused on the equilibriums state investigation. This way, the current analyse triggers the conclusion that adjustment speed of bounded rational player, as well as the differentiation degree, strongly impact the stability of the Nash equilibrium. This dynamic approach and therefore the conclusions of this paper are intented to generate reader's interest from both mathematical and economic point of view.

Keywords: Bertrand model, oligopoly, stability, product differentiation, dynamic approach.
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Oligopoly represents a market scenario where several firms actions impact the aggregate offered output / the selling price of the same or different homogeneous products. More precisely, their mutual interdependences makes oligopolists consider the answer of his competitors at their own actions. In the previous economic literature, many models have been developed, with the special aim to simulate firms behavior in oligopoly / duopoly markets. Two of the most important such models are Cournot model and Bertrand model.

First time presented in Augustine Cournot's book, *„Recherches sur les Principes Mathematics de la Theorie de la Richesse" (1838),* Cournot model describe a duopoly situation, where firms produce homogeneous products, deciding to adopt an output strategy. Although was initially written as a Cournot's theory review, Bertrand's price strategy approach (1883) has become the most used model in price competition scenarios. In both Cournot and Bertrand initial models, the players are enhanced with naive expectations under perfect information, each of them with perfect awareness of his rival's output/price. But in the real economy, it's impossible that all players be naive and information always distributed among them. Hence, the papers conceived in the latest decades, propose different kinds of players' expectations: naive, bounded rational and adaptive (Agiza and Alsadany, 2004).

Changing the direction and referring now at the game type, Augustine Cournot but also Joseph Louis Francois Bertrand studied duopolies where both firms had static expectations regarding his rival's action. The equilibria are globally stable, due to the linearity of the model (cost and demand functions) but also to the small number of firms in the market. Either by increasing the number of competitors, by introducing nonlinearities, or both, a dynamical behaviour may become more interesting. The equilibrium become unstable, multiple equilibria may appear and complex dynamics can be obtained.

Review of the scientific literature

Particulary speaking about the Bertrand duopoly, the introduction of some degree of product differentiation (heterogeneous products) is absolutely necessary in order to avoid the so-called Bertrand paradox (homogenous product unique equilibrium - price matching marginal cost, both players offer half of the existing market output, whilst aggregate profit is zero). This particular scenario rarely appears in practice because the products are almost always differentiated in some way other than price (Dubiel-Teleszynski, 2010; Fanti, et al., 2013).

A wide range of studies based on Bertrand model can be found in the current oligopoly literature. Using general framework previously introduced by Dixit (1979), Singh and Vives (1984) highlight quantity competition as dominant strategy in a substitute products scenario, but also price competition dominance in a complementary product situation. A little bit later, Hackner (2000), Zanchettin (2006) and Tremblay (2011) are making use by a different approach, based on informational asymmetry (including demand's asymmetry) triggering optimality of Bertrand or Cournot-Bertrand models. Regardless the approach, demand and cost function linearity were the common denominator for plenty of studies (Ahmed, et al., 2006; Zhang, et al., 2009; Ma and Sun, 2015; Puu and Tramontana, 2019; Din and Sun, 2020).

Demand function non-linearity was debated by Ahmed, et al. (2015), in the tentative of investigate a dynamic Bertrand duopoly with differentiated goods, where bounded rational players apply a gradient adjustment mechanism in order to update their price in each period.

Ma and Wu (2013) introduced a Bertrand triopoly scenario with bounded rational players, finding out that the time delay may not improve the game stability area. Yi and Zeng (2015) analyzed the air-conditioning chinese market through a dynamic Bertrand duopoly model with quadratic cost function, which is closer to reality but introducing a new perspective versus the existing economic literature. Both studies common hypothesys is the cost function non-linearity.

In the next section we apply the tools used before by Agiza and Elsadany (2003) and Zhang et al (2009) to investigate the dynamics of Bertrand duopoly model with bounded rational players. Assuming that both players offer differentiated products, with price adjustment mechanism based on their last period marginal profit, the common declared aim remain own profit maximisation. We will also present the explicit parametric equations, the conditions of equilibrium points existence and also local stability.

Research methodology

We use a duopoly market scenario, with players charging different prices for their products. Let $p_i(t)$, $i = \overline{1,2}$ represent the price charged by firm i during a certain time period $t = \overline{0, n}$. Each firm sells the quantity q_i, its linearity deriving directly from the inverse demand function expression

$$p_i = a - q_i - dq_j, \quad i,j = \overline{1,2} \qquad (1)$$

where a>0 and d ϵ [0, 1]. The maximal value d=1 leads to a homogeneous products scenario, whilst behavioural patterns of monopolists are reflected in d=0 situation. Hence, d parameter's value is inversely proportional with differentiation degree (if d value increases, differentiation diminishes), also reflecting the nature of the products: positive values highlight substitute products, negatives values complements, whilst zero values reflect independent products. The demand function for a specified product, decreases in its price, but increases/decreases in rival's price, in substitute/complement products scenario.
In our model, the basic system in strategic variables, p_1 and p_2 is:

$$\begin{cases} q_1 = a - p_1 - dq_2 \\ q_2 = a - p_2 - dq_1 \end{cases} \qquad (2)$$

We further assume that firms production costs are linear, different (c_1 and c_2), matching also marginal costs. Accordingly, the profit function form become:

$$\pi_i = (p_i - c_i)q_i\ , (\forall)i = \overline{1,2} \quad (3)$$

Each firm's expected price for t+1 period can be found by solving the optimization problem:

$$\begin{cases} p_1(t+1) = argmax\ \pi_1(p_1(t), p_2^*(t+1)) \\ p_2(t+1) = argmax\ \pi_2(p_1^*(t+1), p_2(t)) \end{cases} \quad (4)$$

where $p_i^*(t+1)$ represents j firm expectation regarding his rival's product price during period t + 1 ($i, j = \overline{1,2},\ i \neq j$).

The *i* firm marginal profit expression in *t* period (preceding calculations in Appendix A) is:

$$\frac{\partial \pi_\mathrm{i}}{\partial p_\mathrm{i}} = \frac{a(1-d)+c_i}{1-d^2} - \frac{2p_\mathrm{i}}{1-d^2} + \frac{\mathrm{d}p_\mathrm{j}}{1-d^2},\ \ i,j = \overline{1,2},\ i \neq j \quad (5)$$

leads to the unique solution of the optimization problem

$$p_\mathrm{i} = \frac{a(1-d)+c_i+\mathrm{d}p_\mathrm{j}}{2} \quad (6)$$

In the real market, the information is usually incomplete, meaning players can use more sophisticated expectations, such as bounded rationality. Decision-making represent an adjustment process based on the of the game last period outcomes. The bounded rational player has no complete knowledge of market, thus he determines production price using the information of local profit maximizers. Consequently, decides to increase / decrease its price if he registered positive / negative marginal profit. Such an adjustment mechanism has been called myopic by Dixit. Hence, the dynamic adjustment mechanism can be modelled as

$$p_i(t+1) = p_i(t) + \alpha_i p_i(t)\frac{\partial \pi_i(p_i, p_j)}{\partial p_i}\ \ i,j = \overline{1,2} \quad (7)$$

where positive parameter α_i represents the adjustment speed of i firm. In these circumstances, a duopoly game with heterogeneous players can be described by using the equational system:

$$\begin{cases} p_1' = p_1 + \alpha_1 p_1\left(\frac{a(1-d)+c_1}{1-d^2} - \frac{2p_1}{1-d^2} + \frac{\mathrm{d}p_2}{1-d^2}\right) \\ p_2' = p_2 + \alpha_2 p_2\left(\frac{a(1-d)+c_2}{1-d^2} - \frac{2p_2}{1-d^2} + \frac{\mathrm{d}p_1}{1-d^2}\right) \end{cases} \quad (8)$$

‘‘ ′ ’’ denotes the unit-time advancement, meaning if the right side variables represents period t results, then the left side ones represent outcomes of period t + 1.

Results and discussion

Due to the fact that Bertrand model is a economic model, we only study the dynamic behavior of the nonnegative equilibrium points (negative variable values almost always are meaningless in economics). Setting p_1➔ p_1' and p_2➔ p_2' , the fixed points are obtained as nonnegative solutions of the non-linear algebric system:

$$\begin{cases} \mathrm{p}_1[a(1-d)+c_1-2\mathrm{p}_1+\mathrm{d}p_2] = 0 \\ \mathrm{p}_2[a(1-d)+c_2-2\mathrm{p}_2+\mathrm{d}p_1] = 0 \end{cases} \quad (9)$$

We found four such points: $\mathrm{E}_1(0,0)$, $\mathrm{E}_2(0, \frac{a(1-d)+c_2}{2})$, $\mathrm{E}_3(\frac{a(1-d)+c_1}{2}, 0)$ and last but not least $\mathrm{E}_4\left(\frac{-\mathrm{ad}(1+\mathrm{d})+2a+\mathrm{d}c_2+2c_1}{4-d^2}, \frac{-ad(1+d)+2a+2c_2+\mathrm{d}c_1}{4-d^2}\right)$ (Appendix B)

Points E_1, E_2, E_3 clearly represents boundary equilibriums. The fixed point E_4 is a Nash equilibrium with full economic meaning, because initial assumption guarantee that $d < 2$ is an always fulfilled must. We are further focusing on the local stability of equilibrium, studying the Jacobian matrix of the sytem (8), considering its eigenvalues in the complex plane study field. The Jacobian matrix at the point (p_1, p_2) has the form:

$$J=\begin{pmatrix} 1+\alpha_1[\frac{a(1-d)+c_1}{1-d^2}-\frac{4p_1}{1-d^2}+\frac{\mathrm{d}p_2}{1-d^2}] & \alpha_1\frac{\mathrm{d}p_1}{1-d^2} \\ \alpha_2\frac{\mathrm{d}p_2}{1-d^2} & 1+\alpha_2[\frac{a(1-d)+c_2}{1-d^2}-\frac{4p_2}{1-d^2}+\frac{\mathrm{d}p_1}{1-d^2}] \end{pmatrix} \quad (10)$$

Proposition 1: The boundary equilibria E_1, E_2, E_3 are unstable equilibrium points.

Proof: In order to prove the above mentioned result, we first determine Jacobian matrix $J(E_1)$:

$$J(E_1)=\begin{pmatrix} 1+\alpha_1\frac{a(1-d)+c_1}{1-d^2} & 0 \\ 0 & 1+\alpha_2\frac{a(1-d)+c_2}{1-d^2} \end{pmatrix} \quad (11)$$

whose eigenvalues are $\boldsymbol{\lambda}_1=1+\alpha_1\frac{a(1-d)+c_1}{1-d^2}, \boldsymbol{\lambda}_2=1+\alpha_2\frac{a(1-d)+c_2}{1-d^2}$

From the a, c_1 and c_2 positivity, also d ϵ [0, 1], result that $|\boldsymbol{\lambda}_1|>1, |\boldsymbol{\lambda}_2|>1$. Then the equilibrium point $E_1(0,0)$ proved to be an unstable node.

Following the same rationality for the second point E_2, the Jacobian matrix $J(E_2)$ will be:

$$J(E_2)=\begin{pmatrix} 1+\alpha_1\left[\frac{a(1-d)+c_1}{1-d^2}+\frac{ad(1-d)+dc_2}{2(1-d^2)}\right] & 0 \\ \alpha_2 d\frac{a(1-d)+c_2}{2} & 1-\alpha_2\frac{a(1-d)+c_2}{1-d^2} \end{pmatrix} \quad (12)$$

This time, the eigenvalues are $\boldsymbol{\lambda}_1=1+\alpha_1\frac{a(2-d-d^2)+2c_1+dc_2}{2(1-d^2)}, \boldsymbol{\lambda}_2=1-\alpha_2\frac{a(1-d)+c_2}{1-d^2}$

Parameters a, c_1, c_2 and d initial hipothesys, leads to the conclusion that $|\boldsymbol{\lambda}_1|>1, |\boldsymbol{\lambda}_2|<1$. Consequentely, $E_2(0,\frac{a(1-d)+c_2}{2})$ represent a saddle point.

In the same manner we treat point E_3; the $J(E_3)$ matrix become:

$$J(E_3)=\begin{pmatrix} 1-\alpha_1\frac{a(1-d)+c_1}{1-d^2} & \alpha_1 d\frac{a(1-d)+c_1}{2} \\ 0 & 1+\alpha_2\left[\frac{a(1-d)+c_2}{1-d^2}+\frac{ad(1-d)+dc_1}{2(1-d^2)}\right] \end{pmatrix} \quad (13)$$

having eigenvalues $\boldsymbol{\lambda}_1=1-\alpha_1\frac{a(1-d)+c_1}{1-d^2}, \boldsymbol{\lambda}_2=1+\alpha_2\frac{a(2-d-d^2)+2c_2+dc_1}{2(1-d^2)}$. Point $E_3(\frac{a(1-d)+c_1}{2},0)$ is a second saddle point.

Presented scenarios transposed in the real economic climate, compell firms to retire from the market. It is an economic anomaly, no firm will accept such a charged price lowering, in order to direct the economic result down to the breakeven point. Thus, the leaving market option could be anytime materialised.

Proposition 2: The Nash equilibrium point E_4 is stable, only if

$$\frac{4}{1-d^2}(\alpha_1 p_1^*+\alpha_2 p_2^*)-4<\alpha_1\alpha_2\frac{(2-\mathrm{d})(2+\mathrm{d})p_1^*p_2^*}{(1-d^2)^2}<\frac{2}{1-d^2}(\alpha_1 p_1^*+\alpha_2 p_2^*)$$

Proof. Finally, we investigate the local stability of equilibrium point E_4. Jacobian matrix is:

$$J(E_4)=\begin{pmatrix} 1-2\alpha_1\frac{p_1^*}{1-d^2} & \alpha_1 d\frac{p_1^*}{1-d^2} \\ \alpha_2 d\frac{p_2^*}{1-d^2} & 1-2\alpha_2\frac{p_2^*}{1-d^2} \end{pmatrix} \quad (14)$$

where $p_1^*=\frac{-\mathrm{ad}(1+\mathrm{d})+2a+\mathrm{d}c_2+2c_1}{4-d^2}$, $p_2^*=\frac{-ad(1+d)+2a+2c_2+\mathrm{d}c_1}{4-d^2}$

The characteristic equation of the $J(E_4)$ has the form $f(\boldsymbol{\lambda})=\boldsymbol{\lambda}^2-Tr\,(J)\boldsymbol{\lambda}+Det\,(J)=0$, where Tr(J) / Det(J) represent Jacobian matrix trace / determinant, described by

$$Tr\,(J) = 2 - \frac{2}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) \qquad (15)$$

$$Det(J) = \;1 - 2\alpha_2 \frac{p_2^*}{1-d^2} - 2\alpha_1 \frac{p_1^*}{1-d^2} + 4\alpha_1\alpha_2 \frac{p_1^* p_2^*}{(1-d^2)^2} - \alpha_1\alpha_2 \frac{d^2\, p_1^* p_2^*}{(1-d^2)^2} \qquad (16)$$

Second order discriminant prove to be positive, meaning that both eigenvalues are real.

$$Tr^{\,2}(J) - 4Det(J) = \;4 - \frac{8}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) + \frac{4}{(1-d^2)^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*)^2 - 4 + 8\alpha_2 \frac{p_2^*}{1-d^2} + 8\alpha_1 \frac{p_1^*}{1-d^2} + 16\alpha_1\alpha_2 \frac{p_1^* p_2^*}{(1-d^2)^2} + 4\alpha_1\alpha_2 \frac{d^2\, p_1^* p_2^*}{(1-d^2)^2} = \frac{4}{(1-d^2)^2}(\,\alpha_1\, p_1^* - \,\alpha_2\, p_2^*)^2 + 4\alpha_1\alpha_2 \frac{d^2\, p_1^* p_2^*}{(1-d^2)^2} > 0 \Rightarrow \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \,\epsilon\, R \qquad (17)$$

The local stability conditions of Nash equilibrium are given by Jury's conditions:

$$\begin{cases} 1 - Tr\,(J) + Det(J) > 0 \\ 1 + Tr\,(J) + Det(J) > 0 \\ \qquad Det(J) < 1 \end{cases} \qquad (18)$$

First one veracity imediately result:

$$1 - Tr\,(J) + Det(J) = \frac{2}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) - \frac{2(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*)}{1-d^2} + 4\alpha_1\alpha_2 \frac{d^2\, p_1^* p_2^*}{(1-d^2)^2} - \alpha_1\alpha_2 \frac{d^2\, p_1^* p_2^*}{(1-d^2)^2} = \alpha_1\alpha_2 \frac{(2-\mathrm{d})(2+\mathrm{d})\, p_1^* p_2^*}{(1-d^2)^2} > 0 \;(A)$$

The last two conditions define a bounded region of stability in the plane of speeds adjustment.

$$1 + Tr\,(J) + Det(J) = 4 - \frac{4}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) + \alpha_1\alpha_2 \frac{(2-\mathrm{d})(2+\mathrm{d}) p_1^* p_2^*}{(1-d^2)^2} > 0$$

$$Det\,(J) - 1 = -\frac{2}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) + \alpha_1\alpha_2 \frac{(2-\mathrm{d})(2+\mathrm{d}) p_1^* p_2^*}{(1-d^2)^2} < 0$$

The necessary and sufficient stability condition is obtained by assembling above information:

$$\frac{4}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) - 4 < \alpha_1\alpha_2 \frac{(2-\mathrm{d})(2+\mathrm{d}) p_1^* p_2^*}{(1-d^2)^2} < \frac{2}{1-d^2}(\,\alpha_1\, p_1^* + \,\alpha_2\, p_2^*) \qquad (19)$$

Certainly, the Nash equilibrium point E_4 is stable in the area defined by Eq. (19) and loose its stability outside this area. In other words, this equation defines a stability region of Nash equilibrium point E_4.

Conclusions

In the present paper, we have studied the dynamics of a repeated Bertrand duopoly model with bounded rational players. The mathematic analyse trigger the existence of four equilibrium points (whose stability is also investigated) highlighting the idea that the adjustment speed in a bounded rationality scenario (parameter α_1 and α_2 values) affect the stability of Nash equilibrium point. Lower values of speeds of adjustment, guarantee a stable Nash equilibrium of the game, in strictly interdependence with the differentiation degree and all other parameters evolution. Adjusting too fast the price speeds, in order to increase own profits, the equilibrium may become unstable and the system will fall into chaos (dynamics complex phenomenons such as bifurcations, attractors and cycles will appear). We have also offer some economic explanations to various dynamic processes in the Bertrand market, providing this way, theoretical references for firms activity.

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References

Agiza, H.N., Elsadany, A.A., 2003. Nonlinear dynamics in the Cournot duopoly game with heterogeneous players. *Phys A*, 320, pp.512–524.

Agiza H.N., Elsadany, A.A., 2004. Chaotic dynamics in nonlinear duopoly game with heterogeneous players. *Applied Mathematics and Computations*, 149, pp.843–860.

Ahmed, E., Elettreby, M.F., Hegazi, A.S., 2006. On Puu's incomplete information formulation for the standard and multi-team bertrand game. *Chaos, Soliton Fractals*, 30, pp.1180–1184.

Ahmed, E., Elsadany, A.A., Puu, T., 2015. On Bertrand duopoly game with differentiated goods. *Applied Mathematics and Computation*, 251, pp.169–179.

Bertrand, J., 1883. Review of Théorie Mathématique de la Richesse Sociale and Recherches sur les Principles Mathématique de Théorie des Richesses. *Journal des Savants*, pp.499–508

Cournot, A., 1838. *Researches into the Mathematical Principles of the Theory of Wealth*. Paris: L. Hachette.

Din, H.R., Sun, C.H., 2020. Welfare improving licensing with endogenous choice of price versus quantities. *North American Journal of Economics and Finance*, 51, Article number: 100859.

Dixit, A., 1979. A model of duopoly suggesting a theory of entry barriers. *Bell Journal of Economics*, 10, pp.20–32.

Dubiel-Teleszynski, T., 2010. Complex dynamics in a Bertrand duopoly game with heterogeneous players. *Central Europe Journal Econ Modell Econometrics*, 2, pp.95–116.

Fanti L, Gori L, Mammana C, Michetti E., 2013. The dynamics of a Bertrand duopoly with differentiated products: synchronization, intermittency and global dynamics. *Chaos Solitons Fractals*, 52, pp.73–86.

Hackner, J., 2000. A note on price and quantity competition in differentiated Oligopolies. *Journal of Economic Theory*, 93, pp.233–239.

Ma, J., Wu, K., 2013. Complex system and influence of delayed decision on the stability of a triopoly price game model. *Nonlinear Dynamics*, 73(3), pp.1741–1751.

Ma, J., Sun, L., 2015. Study and simulation on discrete dynamics of Bertrand triopoly team-game. *Mathematical Problems in Engineering*, 2015, ID 960380, 12 pages.

Puu, T., Tramontana, F., 2019. Can Bertrand and Cournot oligopolies be combined?. *Chaos, Solitons and Fractals*, 125, pp.97-107.

Singh, N., Vives, X., 1984. Price and Quantity Competition in a Differentiated Duopoly. *Rand Journal of Economics*, 15(4), pp.546-554.

Tremblay, C.H., Tremblay, V.J., 2011. The Cournot-Bertrand Model and the Degree of Product Differentiation. *Eco-nomics Letters*, 111(3), pp.233-235.

Yi, Q.G., Zeng, X.J, 2015. Complex dynamics and chaos control of duopoly Bertrand model in Chinese air-conditioning market. *Chaos, Solitons and Fractals*, 76, pp.231-237.

Zanchettin, P., 2006. Differentiated duopoly with asymmetric costs. *Journal of Economic Management Strategy*, 15, pp.999–1015.

Zhang, J., Da, Q., Wang, Y., 2009. The dynamics of Bertrand model with bounded rationality. *Chaos, Solitons Fractals*, 39, pp.2048–2055.

Appendix A

$$\begin{cases} p_1 = a - q_1 - dq_2 \Rightarrow q_1 = a - p_1 - dq_2 \\ p_2 = a - q_2 - dq_1 \Rightarrow q_2 = a - p_2 - dq_1 \end{cases} \rightarrow \; q_1 = a - p_1 - d(a - p_2 - dq_1) = a - p_1 - ad + dp_2 + d^2 q_1$$

$$(1-d^2)q_1 = a(1-d) - p_1 + dp_2 \Rightarrow q_1 = \frac{a(1-d)}{1-d^2} - \frac{p_1}{1-d^2} + \frac{dp_2}{1-d^2}$$

$$q_2 = a - p_2 - \frac{ad(1-d)}{1-d^2} + \frac{dp_1}{1-d^2} - \frac{d^2p_2}{1-d^2}$$

$$q_2 = \frac{a - ad^2 - p_2 + d^2p_2 - ad + ad^2 + dp_1 - d^2p_2}{1-d^2} = \frac{a(1-d)}{1-d^2} - \frac{p_2}{1-d^2} + \frac{\mathrm{d}p_1}{1-d^2}$$

$$\pi_1 = (p_1 - c_1)q_1 = \frac{a(1-d)}{1-d^2}p_1 - \frac{p_1^2}{1-d^2} + \frac{\mathrm{d}p_1p_2}{1-d^2} - \frac{a(1-d)c_1}{1-d^2} + \frac{c_1p_1}{1-d^2} - \frac{\mathrm{d}p_2c_1}{1-d^2}$$

$$\frac{\partial\pi_1}{\partial p_1} = \frac{a(1-d)+c_1}{1-d^2} - \frac{2p_1}{1-d^2} + \frac{\mathrm{d}p_2}{1-d^2} = 0 \Rightarrow p_1 = \frac{a(1-d)+c_1+\mathrm{d}p_2}{2}$$

$$\pi_2 = (p_2 - c_2)q_2 = \frac{a(1-d)}{1-d^2}p_2 - \frac{p_2^2}{1-d^2} + \frac{\mathrm{d}p_1p_2}{1-d^2} - \frac{a(1-d)c_2}{1-d^2} + \frac{c_2p_2}{1-d^2} - \frac{\mathrm{d}p_1c_2}{1-d^2}$$

$$\frac{\partial\pi_2}{\partial p_2} = \frac{a(1-d)+c_2}{1-d^2} - \frac{2p_2}{1-d^2} + \frac{\mathrm{d}p_1}{1-d^2} = 0 \Rightarrow p_2 = \frac{a(1-d)+c_2+\mathrm{d}p_1}{2}$$

Appendix B

If $p_1 \Rightarrow p_1'$ and $p_2 \Rightarrow p_2'$ then $\begin{cases} \mathrm{p}_1[a(1-d)+c_1-2\mathrm{p}_1+\mathrm{d}p_2]=0 \\ \mathrm{p}_2[a(1-d)+c_2-2\mathrm{p}_2+\mathrm{d}p_1]=0 \end{cases}$

a) $\mathrm{p}_1 = 0$, $\mathrm{p}_2 = 0$ $\Rightarrow \mathrm{E}_1(0,0)$

b) $\mathrm{p}_1 = 0$, $a(1-d)+c_2-2\mathrm{p}_2+\mathrm{d}p_1 = 0 \Rightarrow p_2 = \frac{a(1-d)+c_2}{2} \Rightarrow \mathrm{E}_2(0,\frac{a(1-d)+c_2}{2})$

c) $\mathrm{p}_2 = 0$, $a(1-d)+c_1-2\mathrm{p}_1+\mathrm{d}p_2 = 0 \Rightarrow p_1 = \frac{a(1-d)+c_1}{2} \Rightarrow \mathrm{E}_3(\frac{a(1-d)+c_1}{2},0)$

d) $a(1-d)+c_1-2\mathrm{p}_1+\mathrm{d}p_2 = 0$, $a(1-d)+c_2-2\mathrm{p}_2+\mathrm{d}p_1 = 0$

$$\begin{cases} a(1-d)+c_1-2\mathrm{p}_1+\mathrm{d}p_2=0 \\ a(1-d)+c_2-2\mathrm{p}_2+\mathrm{d}p_1=0 \end{cases} \Rightarrow \begin{cases} 2\mathrm{p}_1-\mathrm{d}p_2 = a(1-d)+c_1 & (d) \\ \mathrm{d}p_1-2\mathrm{p}_2 = -a(1-d)-c_2 & (-2) \end{cases}$$

$$\begin{cases} 2d\mathrm{p}_1 - d^2p_2 = ad(1-d)+dc_1 \\ -2\mathrm{d}p_1+4\mathrm{p}_2 = 2a(1-d)+2c_2 \end{cases} \overset{sum}{\Longrightarrow} \mathrm{p}_2(4-d^2) = -ad-ad^2+2a+dc_1+2c_2$$

$$p_2 = \frac{-ad(1+d)+2a+2c_2+\mathrm{d}c_1}{4-d^2} \Rightarrow -2\mathrm{dp}_1 + \frac{-4ad(1+d)+8a+8c_2+4\mathrm{d}c_1}{4-d^2} = 2a(1-d)+2c_2$$

$$-2\mathrm{dp}_1 = \frac{8a(1-d)+8c_2-2ad^2(1-d)-2d^2c_2+4ad(1+d)-8a-4\mathrm{d}c_1-8c_2}{4-d^2}$$

$$-2\mathrm{dp}_1 = \frac{8a-8\mathrm{ad}+8c_2-2ad^2+2ad^3-2d^2c_2+4ad+4ad^2-8a-4\mathrm{d}c_1-8c_2}{4-d^2}$$

$$-2\mathrm{dp}_1 = \frac{-4\mathrm{ad}+2ad^2+2ad^3-2d^2c_2-4\mathrm{d}c_1}{4-d^2} \Rightarrow \mathrm{p}_1 = \frac{-\mathrm{ad}(1+\mathrm{d})+2a+\mathrm{d}c_2+2c_1}{4-d^2}$$

$$\Rightarrow \mathrm{E}_4(p_1^*, p_2^*),\ p_1^* = \frac{-\mathrm{ad}(1+\mathrm{d})+2a+\mathrm{d}c_2+2c_1}{4-d^2} \quad p_2^* = \frac{-ad(1+d)+2a+2c_2+\mathrm{d}c_1}{4-d^2}$$