PRODUCT DIFFERENTIATION INFLUENCE IN THE COURNOT, BERTRAND AND HIERARCHICAL STACKELBERG DUOPOLIES

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Abstract
Imperfect competition is one of the main topics of modern economic analysis and it can be easily distinguished in the current economic climate. Whether we are considering the general economic spectrum or a certain industry, there are a few concepts standing out due to their impact on the tools and methodologies used in maket analysis: price and quantity competition scenarios differences, timing of competitors movement in a duopoly game, role of product differentiation in determining market price/output levels and the presence of the market hierarchical structure. In order to explore the topic (imperfect competition) and, in particular, the impact of the above mentioned concepts in a duopoly market, this paper is using a consistent framework, making use of a product differentiation base model. The paper is deliberating on the market outcomes under price and quantity competition (Cournot and Bertrand simultaneous moves scenarios) but also sequential moves output competition (Stackelberg duopoly). It is also presenting a numerical simulation analyze that can be used to efficiently explore the model properties under the assumption of products differentiation degree variation.

Keywords
Cournot model, Bertrand model, Stackelberg model, oligopoly, stability, product differentiation.

JEL classification
C72, D01, D43, L13

Introduction

One of the main forms of imperfect competition is the oligopoly. During various oligopoly theories over the years, three standard textbook models have been developed in a tentative to explain both the economic output and pricing related decisions: Cournot model (1838), Bertrand model (1883) and last but not the least Stackelberg model (1934). Due to the existence of multiple types of firms interaction and the complex nature of the interdependences between them, the use of only one oligopoly model is not adequate, therefore this paper will consider the three above mentioned models. There are also at least four key aspects which should be considered at the start of any market structure analysis: the decision of competing in output or price terms (very important topic in industrial organization), the timing of the competitors movement (simultaneous or sequential), the products typology (homogeneous or differentiated) as well as the existence / absence of a hierarchical structure. Subject to the above mentioned factors mixture, the market performance and profit distribution will be significantly differred.

In both Cournot or Betrand models, the players are choosing their strategy simultaneously, however whilst Cournot player establish its output level (with price beeing determined by some unspecified agent and triggering market demand to equal the aggregate offer), Bertrand player will focus on the selling price (with firms being constrained to immediately meet the resulting customer demand). Stackelberg model instead is an hierachical model with firms choosing their output level sequentially. The sophisticated firm (the leader) takes into account its ability to manipulate the other firm's output, while the naïve firm (the follower) adopt a Cournot behavior, considering his rival's output level fixed.

The current literature comparing the two output terms equilibria (Cournot and Stackelberg) is abundant. The most common conclusion is that the Stackelberg equilibrium is more efficient than the Cournot equilibrium, total surplus being higher in the sequential game scenario, as per Boyer and Moreaux (1986 and 1987), Daughety (1990); Robson (1990), Albaek (1990), Anderson and Engers, (1992), Amir and Grilo, (1999), Ino and Matsumura, (2012) papers. Other researchers were focused on the direction of simultaneous game scenarios comparison (Cheng, 1985; Judd, 1989; Symeonidis, 2003; Haraguchi and Matsumura,2015; etc). In return, the aggregate analyze including all the three above mentioned models, has received scant attention in the current literature.

The present paper is considering a differentiated products scenario, based on which we are trying to explain Cournot, Bertrand but also Stackelberg static behavior, highlighting some interesting aspects such as firm equilibrium, market surviving potential and the product differentiation impact on Nash equilibrium / subgame perfect equilibrium theory. The originality of this paper consists of the unique approach of bringing together and comparing simultaneously, all three mentioned models, not just theorethically, but also using numerical simulation. The principles of the related mathematic model are also presented below.

The model

The scenario used in this paper is one with plenty consumers but only two producers of differentiated goods. The consumers are targeting to maximize their own satisfaction, described as the difference between own utility function and the necessary spending for purchasing required product amounts (no budgetary constraints are considered):

$$S = U(q_1, q_2) - \sum_{i=1}^{2} p_i q_i \qquad (1)$$

The chosen utility function belongs to quadratic class (non-linear type), having separable variables and beeing also strictly concave (see bellow). The last hypothesis involves double derivability, the existence of the second order derivate and also its negativity.

$$U(q_1, q_2) = aq_1 + aq_2 - \frac{bq_1^2 + 2dq_1q_2 + bq_2^2}{2} \qquad (2)$$

where $a > 0, b > 0,\ d > 0$ (reflecting substitute products). Considering $b > d$, an imperfect substitutability is suggested, whilst setting $b = d$ a homogenous product scenario is assumed.

The starting point in duopoly direct / inverse demand function calculation, is the derivation of the consumer satisfaction function. Their expressions are determinated as follows:

$$\frac{\partial \mathbf{S}}{\partial \mathbf{q_1}} = \mathbf{p_1} = \mathbf{a} - \mathbf{bq_1} - \mathbf{dq_2} \Rightarrow \mathbf{q_1} = \frac{\mathbf{a} - \mathbf{p_1} - \mathbf{dq_2}}{\mathbf{b}} \qquad (3)$$

$$\frac{\partial S}{\partial q_2} = p_2 = a - bq_2 - dq_1 \Rightarrow q_2 = \frac{a - p_2 - dq_1}{b} \qquad (4)$$

Applying substitution methodology, will result:

$$q_1 = \frac{a(b-d) - bp_1 + dp_2}{b^2 - d^2} \quad (5) \qquad q_2 = \frac{a(b-d) - bp_2 + dp_1}{b^2 - d^2} \quad (6)$$

a system similar to those used before by Dixit (1979), Singh andVives (1984), Imperato et all (2004), Tremblay (2011).
It can be noted the necessity that $b > d$ at this stage.

The production cost is further deemed identical for both players, via a linear function (C=c*q), also matching the marginal cost. Based on these assumptions, the profit function become:

$$\pi_i = (p_i - c)q_i \,, (\forall) i = \overline{1,2} \qquad (7)$$

and further

$$\pi_1 = (p_1 - c)q_1 = aq_1 - bq_1^2 - dq_1q_2 - cq_1 \quad (8) \qquad \pi_1 = \frac{a(b-d) - bp_1 + dp_2}{b^2 - d^2}(p_1 - c) \quad (9)$$

$$\pi_2 = (p_2 - c)q_2 = aq_2 - bq_2^2 - dq_1q_2 - cq_2 \quad (10) \qquad \pi_2 = \frac{a(b-d) - bp_2 + dp_1}{b^2 - d^2}(p_2 - c) \quad (11)$$

The market output level / the selling price depends on the two firms interaction type. If the duopolists choose to adopt an output strategy, deciding to take decisions simultaneously, without knowing his rival answer, we face a Cournot behaviour . By solving the profit maximization problem in output terms, we can find out the best response functions. Further application of substituting method leads to the Nash equilibrium output values:

$$\begin{cases} \frac{\partial \pi_1}{\partial q_1} = a - 2bp_1 - dp_2 - c = 0 \\ \frac{\partial \pi_2}{\partial q_2} = a - 2bp_2 - dp_1 - c = 0 \end{cases} \Rightarrow \begin{cases} q_1 = \frac{a-c-dq_2}{2b} \\ q_2 = \frac{a-c-dq_1}{2b} \end{cases} \Rightarrow q_1 = \frac{a-c-d\frac{a-c-dq_1}{2b}}{2b} = \frac{2ab-2bc-ad+cd+d^2q_1}{4b^2}$$

$$q_1(4b^2 - d^2) = a(2b-d) - c(2b-d) \Rightarrow q_1 = \frac{a-c}{2b+d} \; (12) \quad \Rightarrow q_2 = \frac{a-c-d\frac{a-c}{2b+d}}{2b} = \frac{a-c}{2b+d} \quad (13)$$

The corresponding prices, will be immediately determined:

$$p_1 = p_2 = a - (b+d)\frac{a-c}{2b+d} = \frac{ab - bc + c(2b+d)}{2b+d} = c + \frac{b(a-c)}{2b+d} \qquad (14)$$

and finally the profits level become

$$\pi_1 = \pi_2 = (p - c)q = \frac{b(a-c)}{2b+d}\frac{a-c}{2b+d} = \frac{b(a-c)^2}{(2b+d)^2} \qquad (15)$$

If the duopolists decide to compete in price terms instead, their action path beeing also simultaneously manifested, the Bertrand scenario is revealed . Profit’s first order conditions represent the starting point in the determination of the Nash equilibrium price:

$$\begin{cases}\frac{\partial\pi_1}{\partial p_1}=\frac{a(b-d)-bp_1+dp_2}{b^2-d^2}-\frac{b(p_1-c)}{b^2-d^2}=0\\ \frac{\partial\pi_2}{\partial p_2}=\frac{a(b-d)-bp_2+dp_1}{b^2-d^2}-\frac{b(p_2-c)}{b^2-d^2}=0\end{cases}\Rightarrow\begin{cases}2b\,p_1-dp_2=a(b-d)+bc\\ -dp_1+2bp_2=a(b-d)+bc\end{cases}$$

The equations system solution is $p_1 = p_2 = c + \frac{(a-c)(b-d)}{2b-d}$ (16)

By substituting and solving the new equations system (q_1 and q_2 as unknowns), will obtain

$$q_1 = q_2 = \frac{(a-c)(b^2-bd)}{(b^2-d^2)(2b-d)} \quad (17)$$

$$\pi_1 = \pi_2 = (p-c)q = \frac{(a-c)(b-d)}{2b-d}\,\frac{(a-c)(b^2-bd)}{(b^2-d^2)(2b-d)} = \frac{b(a-c)^2(b-d)^2}{(b^2-d^2)(2b-d)^2} \quad (18)$$

If the firms sequential moving scenario is preffered to the simultaneous moving one, we are dealing with a Stackelberg model. Assuming player 1 will move first, the main problem will be to maximize its profit level, considering the subsequent move of his rival, which is not controlable but at list predictable. This aspect can be solved by using backward induction method. In the second stage, the follower chooses an output level to maximize profits given the output choice of the leader. In the first stage instead, first mover chooses its profit maximizing output knowing how his rival will respond. All the mathematic Appendix calculations, leads to the bellow mentioned subgame perfect equilibrium values:

$$q_1 = \frac{(a-c)(2b-d)}{4b^2-2d^2} \quad (19) \qquad q_2 = \frac{(a-c)[2b(2b-d)-d^2]}{2b(4b^2-2d^2)} \quad (20)$$

$$p_1 = a - \frac{(a-c)(2b+d)}{4b} \quad (21) \qquad p_2 = a - \frac{(a-c)[2b(2b+d)-3d^2]}{4(2b^2-d^2)} \quad (22)$$

$$\pi_1 = \frac{(a-c)^2(2b-d)^2}{8b(2b^2-d^2)} \quad (23) \qquad \pi_2 = \frac{(a-c)^2(4b^2-2bd-3d^2)^2}{16b(2b^2-d^2)^2} \quad (24)$$

Comparing the equilibrium values of both output strategy games (Cournot and Stackelberg), we can reach the following conclusions:

- The leader's output level is higher in the sequential game, knowing that the follower will respond by cutting its own;
- The leader/the follower charge lower prices than in simultaneous moving game.
- Although aggregate profits fall, the leader win extra profits by taking a greater market share. This is the first-mover well – known advantage in the Stackelberg game.

Next paragraphs will analyze the $b = d$ situation - perfectly substitutes products scenario. The equilibrium values can be synthesized in (Table no.1):

Table no. 1 Cournot/Bertrand/Stackelberg homogenous products equilibrium figures

Strategic variable	p_1	p_2	q_1	q_2	π_1	π_2
Cournot model	$\frac{a+2c}{3}$	$\frac{a+2c}{3}$	$\frac{a-c}{3b}$	$\frac{a-c}{3b}$	$\frac{(a-c)^2}{9b}$	$\frac{(a-c)^2}{9b}$
Bertrand model	c	c	$\frac{a-c}{2b}$	$\frac{a-c}{2b}$	0	0
Stackelberg model	$\frac{a+3c}{4}$	$\frac{a+3c}{4}$	$\frac{a-c}{2b}$	$\frac{a-c}{4b}$	$\frac{(a-c)^2}{8b}$	$\frac{(a-c)^2}{16b}$

Source: authors' calculations

Remarks:

- The highest output level is reached in the Bertrand game, matching also the leader's output in the sequential model. In the simultaneous output scenario three quarters of this amount is produced, whilst the Stackelberg follower registered half of it.
- Cournot output strategy triggers the highest level of charged price with sequential strategy following closely. The Bertrand game instead, reveals the most interesting view with price matching lowest possible value for maintaining economic rentability, therefore the marginal cost (Bertrand paradox)
- Although Stackelberg's leader attain the highest profit level, aggregate value is maximized in the Cournot simultaneous game. As a consequence of Bertrand paradox, price strategy offers aggregate zero profit.

Simulation study case

The next paragraphs are trying to facilitate even a more detailed understanding of the three behavioral types noted by the paper, as products substitutability degree starts to change. With this precise goal, we exemplify a differentiated products duopoly scenario, assuming specific values for model entrance parameters, as follows: a = 500, b = 3, c = 60. With this hypothesis being made, we consider the main parameter of the model matching d = 2 value.

The demand functions will be obtained by substituting parameters values in (3) and (4):

- $\frac{\partial U}{\partial q_1} = \mathbf{500 - 3q_1 - 2q_2 = p_1} \qquad \frac{\partial U}{\partial q_2} = \mathbf{500 - 3q_2 - 2q_1 = p_2}$

$$q_1 = \frac{500 - p_1 - 2q_2}{3} \qquad q_2 = \frac{500 - p_2 - 2q_1}{3}$$

Further substituting quantities in (5) and (6), inverse demand functions are revealed:

$$q_1 = \frac{500 - p_1 - 2\frac{500 - p_2 - 2q_1}{3}}{3} \rightarrow q_1 = 100 - 0.6p_1 + 0.4p_2$$

$$q_2 = \frac{500 - p_2 - 2\frac{500 - p_1 - 2q_2}{3}}{3} \rightarrow q_2 = 100 - 0.6p_2 + 0.4p_1$$

Consumer surplus can be determinate as the difference between own utility function and price for purchasing required product quantities, whilst total surplus includes also the producers profits.

A complete situation with all equilibrium scenarios values, is presented in (Table no.2):

Table no. 2 Cournot/Bertrand/Stackelberg equilibrium figures in b=3 & d=2 scenario

Strategic variable	p_1	p_2	q_1	q_2	π_1	π_2	S_c	S_t
Cournot model	225	225	55	55	9075	9075	15125	33275
Bertrand model	170	170	66	66	7260	7260	21780	36300
Stackelberg model	206.7	217.1	62.9	52.4	9219.0	8231.3	16627.2	34077.6

Source: authors' calculations

The degree of product differentiation can be easily modified, and we consider now a much differentiated products scenario, as d = 0.5; proper substitutions led to the follow result:

- $\frac{\partial U}{\partial q_1} = \mathbf{500 - 3q_1 - 0.5q_2 = p_1} \qquad \frac{\partial U}{\partial q_2} = \mathbf{500 - 3q_2 - 0.5q_1 = p_2}$

$$q_1 = \frac{500 - p_1 - 0.5q_2}{3} \qquad q_2 = \frac{500 - p_2 - 0.5q_1}{3}$$

whilst inverse demand functions became:

$$q_1 = \frac{500 - p_1 - 2\frac{500 - p_2 - 0.5q_1}{3}}{3} \rightarrow q_1 = 62{,}5 - 0.375p_1 + 0.25p_2$$

$$q_2 = \frac{500 - p_2 - 2\frac{500 - p_1 - 0.5q_2}{3}}{3} \rightarrow q_2 = 62.5 - 0.375p_2 + 0.25p_1$$

Consumer/total surplus and all other equilibrium values, can be observed in (Table no.3):

Table no. 3 Cournot/Bertrand/Stackelberg equilibrium figures in b=3 & d=0.5 scenario

Strategic variable	p_1	p_2	q_1	q_2	π_1	π_2	S_c	S_t
Cournot	263.1	263.1	67.7	67.7	13746.7	13746.7	16037.9	43531.4
Bertrand	260	260	68.6	68.6	13714.3	13714.3	16457.1	43885.7
Stackelberg	261.7	263.0	68.2	67.7	13747.4	13730.6	16141.7	43619.8

Source: authors' calculations

Finally, we treat a high homogeneity product degree scenario, reflected by a d = 2.5 value. Demand function and also his inverse expression will be:

- $\frac{\partial U}{\partial q_1} = 500 - 3q_1 - 2.5q_2 = p_1$ $\quad \frac{\partial U}{\partial q_2} = 500 - 3q_2 - 2.5q_1 = p_2$

$$q_1 = \frac{500 - p_1 - 2.5q_2}{3} \qquad q_2 = \frac{500 - p_2 - 2.5q_1}{3}$$

whilst inverse demand functions became:

$$q_1 = \frac{500 - p_1 - 2\frac{500 - p_2 - 2.5q_1}{3}}{3} \rightarrow q_1 = 125 - 0.75p_1 + 0.5p_2$$

$$q_2 = \frac{500 - p_2 - 2\frac{500 - p_1 - 2.5q_2}{3}}{3} \rightarrow q_2 = 125 - 0.75p_2 + 0.5p_1$$

All the equilibrium values, can be seen in (Table no.4):

Table no. 4 Cournot/Bertrand/Stackelberg equilibrium figures in b=3 & d=2.5 scenario

Strategic variable	p_1	p_2	q_1	q_2	π_1	π_2	S_c	S_t
Cournot model	215.3	215.3	51.8	51.8	8038.8	8038.8	14737.7	30815.2
Bertrand model	122.9	122.9	68.6	68.6	4310.2	4310.2	25861.2	34481.6
Stackelberg model	188.3	198.1	65.5	46.0	8409.9	6355.8	17160.4	31926.1

Source: authors' calculations

Conclusions

In all three models *d* parameter reflects the degree of product differentiation.

Lower positive values highlight a very poor substitutability, the equilibrium values approaching to the the monopoly solution. Once monopoly scenario triggered (zero d parameter value reached), the connection between firms will be lost, and it will no longer matter which strategy will be choosed.

As the *d* value increases, getting close to the *b* parameter value (homogenous products scenario), the results are significantly different. Models based on an output strategy offer a price equilibrium solution, far enough above marginal cost, despite the fact that the profits fall. In an Bertrand game instead, the outcome level approaches marginal cost pricing.

Previous arguments can be easily used in the tentative of making the firms aware of the product differentiation importance, especially in the case of price competition.
Turning our attention over to the Stackelberg sequential model, the paper is trying to explain what exactly happens with the leader's strategical choice when the product differentiation degree changes. In a homogeneous products scenario, the leader decision is to produce an output level matching the monopoly output level. If d parameter start to decrease, the leader will be forced to reduce his produced quantity. As is well known, a small change of product differentiation degree will impact both firms best reply functions. Focusing on the leader, two opposite effects will manifest. Althought for any given level of follower's output, the leader wants to produce more, an increase in the rival's produced amount compell the leader to produce less. The latter effect dominates first, but at a certain moment, the former effect must take over, and the produced amount trend will be inversed, rising toward the monopoly level, as d parameter value approaches to zero.
We can further expand by using math principles to also prove the previously reached conclusions. Focusing on this approach, we are highlighting the above simulation figures, their connection with the monotony of equilibrium values functions, induced by d parameter variation. Thus, last results can be rephrased as follows:

- In a simultaneous output competition game (Cournot game), initial scenario (b=3 & d=2) ofer better results than in the products differentiation degree decreasing scenario, but worse than in the poor substitutability one. Equilibrium price, quantity and also profits follow decreasing trends, as the homogenity degree starts to increase, from the no firm connection case, till the perfect substitutes situation (first order derivates are negatives for all [0;3] interval. Consumer satisfaction is affected in the same way and obviously the aggregate market surplus follows the same trend.
- In the price competition game (Betrand game), different trends are revealed: prices, profits and aggregate market surplus fall as the degree of product differentiation decreases (b=3 & d=0.5 scenario offer the best result). The equilibrium quantity „hide" a scenarios mixture, decreasing trend being valid as long as d value keeps lower than 1,5 (the unique critical point). Once over this value, the trend is inversed, a perfect coefficients distribution symmetry being noted in [0;3] interval. Consumer surplus highlight an increasing trend, the total amount they have to pay for purchasing the desired product quantity, lowering as the product homogeneity increase.
- In the sequential Stackelberg game, prices and profits fall as the substitutability degree increases, as well as the follower produced amount level (second scenario – the best whilst third scenario - the worst). The leader's equilibrium output level instead shows a decreasing trend, starting with initial monopoly situation, as d parameter does not exceed 1,78 value; after that, once the increase begins to manifest, the output level bounce back to the monopoly level. Consumer surplus level registers an increasing trend on the entire studied area, but not good enough to modify the aggregate surplus evolution, strongly influenced by the decrease in the producers profits, once the products differentiation degree has started to increase.

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Appendix

$$\left.\begin{array}{r} a-2bp_2-\mathrm{d}p_1-c=0 \Rightarrow q_2=\frac{a-c-dq_1}{2b} \\ \pi_1=aq_1-b\mathrm{q}_1^2-dq_1q_2-cq_1 \end{array}\right\} \Rightarrow \pi_1=aq_1-b\mathrm{q}_1^2-dq_1\frac{a-c-dq_1}{2b}-cq_1$$

$$2b\pi_1=2abq_1-2b^2\mathrm{q}_1^2-adq_1+cdq_1+d^2\mathrm{q}_1^2-2bcq_1 \Rightarrow \pi_1 = q_1\frac{(a-c)(2b-d)}{2\mathrm{b}}-\frac{2b^2-d^2}{2\mathrm{b}}\mathrm{q}_1^2$$

$$\frac{\partial\pi_1}{\partial q_1}=0 \Rightarrow \frac{(a-c)(2b-d)}{2\mathrm{b}}-\frac{4b^2-2d^2}{2\mathrm{b}}q_1=0 \Rightarrow q_1=\frac{(a-c)(2b-d)}{4b^2-2d^2}$$

$$q_2=\frac{a-c-d\frac{(a-c)(2b-d)}{4b^2-2d^2}}{2b}=\frac{4ab^2-2ad^2-4b^2c+2cd^2-2abd+2bcd+ad^2-cd^2}{2b(4b^2-2d^2)}$$

$$=\frac{4b^2(a-c)-2bd(a-c)-ad^2+cd^2}{2b(4b^2-2d^2)} \Rightarrow q_2=\frac{(a-c)[2b(2b-d)-d^2]}{4b^2-2d^2}$$

$$\mathrm{p}_1=\mathrm{a}-\mathrm{bq}_1-\mathrm{dq}_2=a-\frac{(a-c)[2b^2(2b+d)-d^2(2\mathrm{b}+\mathrm{d})]}{2b(4b^2-2d^2)}=a-\frac{(a-c)(2b+d)}{4b}$$

$$\mathrm{p}_2=\mathrm{a}-\mathrm{bq}_2-\mathrm{dq}_1===a-\frac{(a-c)[2b^2(2b+d)-3\mathrm{bd}^2]}{4b(2b^2-d^2)}=a-\frac{(a-c)[2\mathrm{b}(2b+d)-3d^2]}{4(2b^2-d^2)}$$

$$\pi_1=(p_1-c)q_1==(a-c)\left(1-\frac{2b+d}{4\mathrm{b}}\right)\frac{(a-c)(2b-d)}{4b^2-2d^2}=\frac{(a-c)^2(2b-d)^2}{8b(2b^2-d^2)}$$

$$\pi_2=(p_2-c)q_2=\left\{a-c-\frac{(a-c)[2b(2b+d)-3d^2]}{4(2b^2-d^2)}\right\}\frac{(a-c)[2b(2b-d)-d^2]}{4b^2-2d^2}=$$

$$=(a-c)^2\frac{4b^2-2bd-3d^2}{4(2b^2-d^2)}\,\frac{4b^2-2bd-3d^2}{2b(4b^2-2d^2)}=\frac{(a-c)^2(4b^2-2bd-3d^2)^2}{16b(2b^2-d^2)}$$